Density modulo 1 of sublacunary sequences: application of Peres-Schlag's arguments

Moshchevitin N.G. [1]

Abstract. Let the sequence $\{t_n\}_{n=1}^{\infty}$ of reals satisfy the condition $\frac{t_{n+1}}{t_n} \geqslant 1+\frac{\gamma}{n^\beta}$, $0 \leqslant \beta < 1$, $\gamma > 0$. Then the set $\{\ \alpha \in [0,1]:\ \ \exists \varkappa > 0\ \forall n \in \mathbb{N}\ ||t_n\alpha|| > \frac{\varkappa}{n^\beta \log(n+1)}\ \}$ is uncountable. Moreover its Hausdorff dimension is equal to 1. Consider the set of naturals of the form 2^n3^m and let the sequence $s_1{=}1, s_2{=}2, s_3{=}3, s_4{=}4, s_5{=}6, s_6 = 8, \dots$ performs this set as an increasing sequence. Then the set $\{\ \alpha \in [0,1]:\ \ \exists \varkappa > 0\ \forall n \in \mathbb{N}\ ||s_n\alpha|| > \frac{\varkappa}{\sqrt{n}\log(n+1)}\ \}$ also has Hausdorff dimension equal to 1. The results obtained use an original approach due to Y. Peres and W. Schlag.

1. Introduction. A sequence $\{t_j\}$, $j = 1, 2, 3, ..$ of positive real numbers is defined to be lacunary if for some $M > 0$ one has

$$\frac{t_{j+1}}{t_j} \geqslant 1 + \frac{1}{M}, \ \ \forall j \in \mathbb{N}.$$

Erdös [1] conjectured that for any lacunary sequence there exists real α such that the set of fractional parts $\{\alpha t_j\}$, $j \in \mathbb{N}$ is not dense in $[0,1]$. This conjecture was proved by A. Pollington [2] and B. de Mathan [3]. Some quantitative improvements were due to Y. Katznelson [4], R. Akhunzhanov and N. Moshchevitin [5] and A. Dubickas [6]. The best known quantitative estimate is due to Y. Peres and W. Schlag [7]. The last authors proved that with some positive constant $\gamma > 0$ for any sequence $\{t_j\}$ under consideration there exists a real number α such that

$$||\alpha t_j|| \geqslant \frac{\gamma}{M\log M}, \ \ \forall j \in \mathbb{N}.$$

Y. Peres and W. Schlag use an original approach connected with the Lovasz local lemma.

From another hand R. Akhunzhanov and N. Moshchevitin in [8] generalized Pollington - de Mathan's result to sublacunary sequences. For example for a sequence $\{t_j\}$ under condition

$$\frac{t_{j+1}}{t_j} \geqslant 1 + \frac{\gamma}{n^\beta}, \ \ \forall j \in \mathbb{N}, \gamma > 0, \beta \in (0, 1/2]$$

they proved the existence of real irrational α such that

$$\liminf_{n\to\infty} \left(||t_n\alpha|| \times n^{2\beta}\right) > 0.$$

Another application from [8] deals with the sequence of naturals of the form 2^m3^n, $m, n \in \mathbb{N} \cup \{0\}$.

In the present paper we apply the arguments from [7] to improve the results from [8] mentioned above.

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2. Results. Let $1 \leqslant t_1 < t_2 < ...t_n < t_{n+1} < ...$ be a strictly increasing sequence of reals and $\lim_{n\to\infty} t_n = +\infty$. For a given sequence $\{t_n\}$ we define the function

$$H(n,\tau) = \min\left\{k \in \mathbb{N} : \ \frac{t_{n+k}}{t_n} \geqslant \tau\right\}. \tag{1}$$

Theorem 1. *Let $0 < \eta < 1$. Consider a sequence $\{h(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ of natural numbers such that for all natural n under condition $n > h(n)$ the function $n \mapsto n - h(n)$ is increasing and a decreasing sequence $\{\delta(n)\}_{n=1}^{\infty}$ of positive real numbers. Let the sequence $\{n_k\}_{k=0}^{K}$ of natural numbers is defined to satisfy the condition*

$$n_k = n_{k+1} - h(n_{k+1}) \tag{2}$$

for $0 \leqslant k \leqslant K-1$. Let our sequences satisfy the following conditions (i),(ii) and (iii) below.

(i) For any natural n under condition $n > h(n)$ the following inequality is valid

$$h(n) \geqslant H(n - h(n), 1/\delta(n - h(n))).$$

(ii) For any $k \leqslant K-1$ the following inequality is valid

$$\sum_{v=n_k+1}^{n_{k+1}-1} \delta(v) \leqslant \frac{(1-\eta)\eta}{4}.$$

(iii) For $k = 0$ the following inequality is valid

$$\sum_{v=1}^{n_0} \delta(v) \leqslant \frac{1-\eta}{16}.$$

Then for the set

$$\mathcal{A}_K = \{\alpha \in [0,1] : \ ||t_n\alpha|| > \delta(n) \ \forall n \leqslant n_K\}$$

one has

$$\mu(\mathcal{A}_K) \geqslant \eta^{K+1}.$$

Here $\mu(\cdot)$ denotes the Lebesgue measure. Note that the sets $\mathcal{A}_K$ are closed and nested: $\mathcal{A}_{K+1} \subseteq \mathcal{A}_K$. Moreover if we have a natural number N we can construct a sequence $\{n_k\}$ such that $n_K = N$, the equalities (2) are satisfied, $n_0 = n_1 - h(n_1) \geqslant 1$ but $n_0 - h(n_0) \leqslant 0$. Hence as a corollary of Theorem 1 we immediately obtain

Theorem 2. *Let $0 < \eta < 1$. Consider a sequence $\{h(n)\}_{n=1}^{\infty} \subset \mathbb{N}$ of natural numbers such that for all natural n under condition $n > h(n)$ the function $n \mapsto n - h(n)$ is increasing and a decreasing sequence $\{\delta(n)\}_{n=1}^{\infty}$ of positive reals. Let these sequences satisfy the following conditions (i) from Theorem 1 and the conditions (ii ') (iii') below.*

(ii ') For all natural numbers n under condition $n > h(n)$ the following inequality is valid

$$\sum_{v=n-h(n)+1}^{n-1} \delta(v) \leqslant \frac{(1-\eta)\eta}{4}.$$

(iii ') For all natural numbers n under condition $n \leqslant h(n)$ the following inequality is valid

$$\sum_{v=1}^{n} \delta(v) \leqslant \frac{1-\eta}{16}.$$

Then the set

$$\mathcal{A} = \{\alpha \in [0,1] : \ \ ||t_n\alpha|| > \delta(n) \ \forall n \in \mathbb{N}\}$$

is nonempty.

Theorem 3. *Let the conditions of theorem 2 be satisfied and an infinite sequence $\{n_k\}_{k=0}^{\infty}$ of naturals satisfies the condition (2) for all natural k. Let the series*

$$\sum_{k=1}^{\infty} \frac{1}{\eta^k} \cdot \left(\frac{t_{n_k}}{\delta(n_k)}\right)^{\nu} / \left(\frac{t_{n_{k-1}}}{\delta(n_{k-1})}\right) \tag{3}$$

converges for all $\nu < \nu_0$ Then the set $\mathcal{A}$ from Theorem 2 has Hausdorff dimension $\geqslant \nu_0$.

We give a complete proof of theorem 1 in Sections 3,4. In Section 5 we give comments to the proof of Theorem 3. In section 6 we give some applications of our results.

4. Lemmata. For $n \geqslant 1$ we define

$$l_n = \left\lfloor \log_2 \left(\frac{t_n}{2\delta(n)}\right)\right\rfloor. \tag{4}$$

From monotonicity of t_n and $\delta(n)$ it follows that $l_{n+1} \geqslant l_n$. Put

$$E(n,a) = \left[\frac{a}{t_n} - \frac{\delta(n)}{t_n}, \frac{a}{t_n} + \frac{\delta(n)}{t_n}\right]$$

Let A_n be the union of dyadic intervals of the form

$$\left(\frac{b}{2^{l_n}}, \frac{b+\varepsilon}{2^{l_n}}\right), \ \ b \in \mathbb{Z}, \ \varepsilon \in \{1,2\}$$

which covers the set

$$\bigcup_{0\leqslant a \leqslant \lceil t_n \rceil} E(n,a) \bigcap [0,1].$$

So

$$\bigcup_{0\leqslant a \leqslant \lceil t_n \rceil} E(n,a) \bigcap [0,1] \subseteq A_n.$$

Define $A_n^c = [0,1] \setminus A_n$. Note that

$$\mu(A_n) \leqslant (\lceil t_n \rceil + 1)\frac{2\delta(n)}{t_n} \leqslant 16\delta(n)$$

and

$$\mu\left(\bigcap_{n\leqslant n_0} A_n^c\right) \geqslant 1 - 16\sum_{n=1}^{n_0} \delta(n). \tag{5}$$

Lemma 1. *Let $n > h(n)$. Let the condition (i) holds and*

$$\mu\left(\bigcap_{j\leqslant n-h(n)} A_j^c\right) > 0.$$

Then

$$\mu\left(\bigcap_{j\leqslant n-h(n)} A_j^c \bigcap A_n\right) \leqslant 4\delta(n)\mu\left(\bigcap_{j\leqslant n-h(n)} A_j^c\right). \tag{6}$$

Proof. The set $\bigcap_{j\leqslant n-h(n)} A_j^c$ can be considered as a union

$$\bigcap_{j\leqslant n-h(n)} A_j^c = \bigcup_{\nu=1}^{T} I_\nu \tag{7}$$

of the dyadic intervals $I_\nu = I_\nu^{(n-h(n))}$ of the form

$$\left[\frac{b}{2^{l_{n-h(n)}}}, \frac{b+1}{2^{l_{n-h(n)}}}\right], \ \ b \in \mathbb{Z}$$

where $T \geqslant 1$. Now the set $A_n \cap I_\nu$ can be represented as a union

$$A_n \bigcap I_\nu = \bigcup_{i=1}^{W_\nu} J_i$$

of intervals J_i of the form

$$\left[\frac{b}{2^{l_n)}}, \frac{b+1}{2^{l_n}}\right].$$

Moreover

$$W_\nu \leqslant \left\lfloor \left(\frac{1}{2^{l_{n-h(n)}}} + \frac{\delta(n)}{2^{l_n}}\right) t_n \right\rfloor + 1 \leqslant \frac{t_n}{2^{l_{n-h(n)}}} + 2.$$

So

$$\mu\left(A_n \cap I_\nu\right) = \frac{W_\nu}{2^{l_n}}$$

and

$$\mu\left(\bigcap_{j\leqslant n-h(n)} A_j^c \bigcap A_n\right) \leqslant \frac{T}{2^{l_n}}\left(\frac{t_n}{2^{l_{n-h(n)}}}+2\right) = \mu\left(\bigcap_{j\leqslant n-h(n)} A_j^c\right)\frac{2^{l_{n-h(n)}}}{2^{l_n}}\left(\frac{t_n}{2^{l_{n-h(n)}}}+2\right) =$$

$$= \mu\left(\bigcap_{j\leqslant n-h(n)} A_j^c\right)\left(\frac{t_n}{2^{l_n}} + 2\cdot\frac{2^{l_{n-h(n)}}}{2^{l_n}}\right).$$

But

$$\frac{t_n}{2^{l_n}} \leqslant 2\delta(n) \tag{8}$$

from the definition of l_n (formula (4)). For the second summand we have

$$\frac{2^{l_{n-h(n)}}}{2^{l_n}} \leqslant 2\cdot\frac{t_{n-h(n)}}{t_n}\cdot\frac{\delta(n)}{\delta(n-h(n))} \leqslant 2\delta(n) \tag{9}$$

from the condition (i) and the definition (1) of the function $H(\cdot,\cdot)$.

Now Lemma 1 follows from (8,9).

For fixed τ and $0\leqslant v\leqslant h(\tau)$ define $\tau_v = \tau - h(\tau) + v$. Note that $\tau_{h(\tau)} = \tau$ and $\tau_0 = \tau - h(\tau)$. Note that $\tau_0 \leqslant \tau_v \leqslant \tau$.

Lemma 2. *Let the function $n-h(n)$ is increasing and the condition (i) holds. Let for $\tau_0 > h(\tau_0)$ the following inequality is valid:*

$$\mu\left(\bigcap_{j\leqslant\tau_0} A_j^c\right) \geqslant \eta\mu\left(\bigcap_{j\leqslant\tau_0-h(\tau_0)} A_j^c\right) > 0 \tag{10}$$

with some positive η.

Then we have

$$\mu\left(\bigcap_{j\leqslant\tau} A_j^c\right) \geqslant \left(1-\frac{4}{\eta}\sum_{v=\tau_1}^{\tau-1}\delta(v)\right)\times\mu\left(\bigcap_{j\leqslant\tau_0} A_j^c\right). \tag{11}$$

Proof.

We have

$$\mu\left(\bigcap_{j\leqslant\tau} A_j^c\right) = \mu\left(\left(\cdots\left(\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right)\setminus A_{\tau-h(\tau)+1}\right)\setminus\cdots\right)\setminus A_\tau\right) \geqslant$$

$$\geqslant \mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right) - \sum_{v=1}^{h(\tau)}\mu\left(A_{\tau_v}\bigcap\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right)\right).$$

But as $\tau_v \leqslant \tau$ from the monotonicity condition for $n-h(n)$ we get $\tau-h(\tau)\geqslant\tau_v-h(\tau_v)$ so

$$\bigcap_{j\leqslant\tau-h(\tau)} A_j^c \subseteq \bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c. \tag{12}$$

Now

$$\mu\left(\bigcap_{j\leqslant\tau} A_j^c\right) \geqslant \mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right) - \sum_{v=1}^{h(\tau)}\mu\left(A_{\tau_v}\bigcap\left(\bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c\right)\right).$$

We apply Lemma 1 for $n=\tau_v,\ v=1,...,h(\tau)$ (it is possible as from (12) and $\mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right)>0$ it follows that $\mu\left(\bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c\right)>0$ for all v) and obtain the inequality

$$\mu\left(A_{\tau_v}\bigcap\left(\bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c\right)\right) \leqslant 4\delta(\tau_v)\mu\left(\bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c\right).$$

Now

$$\mu\left(\bigcap_{j\leqslant\tau} A_j^c\right) \geqslant \mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right) - 4\left(\sum_{v=1}^{h(\tau)}\delta(\tau_v)\right)\times\max_{1\leqslant v<h(\tau)}\mu\left(\bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c\right) \geqslant$$

$$\mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right) - 4\left(\sum_{v=1}^{h(\tau)}\delta(\tau_v)\right)\times\max_{0\leqslant v<h(\tau)}\mu\left(\bigcap_{j\leqslant\tau_v-h(\tau_v)} A_j^c\right).$$

But we have the condition that the function $n-h(n)$ is increasing. So the maximum here is obtained at $v=0$. It follows that

$$\mu\left(\bigcap_{j\leqslant\tau} A_j^c\right) \geqslant \mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right) - 4\left(\sum_{v=1}^{h(\tau)}\delta(\tau_v)\right)\times\mu\left(\bigcap_{j\leqslant\tau_0-h(\tau_0)} A_j^c\right).$$

We apply (10) below:

$$\mu\left(\bigcap_{j\leqslant\tau} A_j^c\right) \geqslant \mu\left(\bigcap_{j\leqslant\tau-h(\tau)} A_j^c\right) - \frac{4}{\eta}\left(\sum_{v=1}^{h(\tau)}\delta(\tau_v)\right)\times\mu\left(\bigcap_{j\leqslant\tau_0} A_j^c\right).$$

Remember that $\tau_0 = \tau - h(\tau)$ and Lemma 2 follows.

4. Proof of Theorem 1. From condition (iii) of the Theorem 1 and (5) it follows that $\mu\left(\bigcap_{j\leqslant n_0} A_j^c\right) \geqslant \eta \geqslant \eta\mu\left(\bigcap_{j\leqslant n_0-h(n_0)} A_j^c\right)$. This is the base of induction. The inductive step $\mu\left(\bigcap_{j\leqslant n_{k+1}} A_j^c\right) \geqslant \eta\mu\left(\bigcap_{j\leqslant n_k} A_j^c\right)$ follows from condition (ii) and Lemma 2: We must put $\tau = n_{k+1}$, then $\tau_0 = n_k$. From inductive hypothesis we have (10). The condition (ii) leads to inequality $1 - \frac{4}{\eta}\sum_{v=\tau_1}^{\tau-1}\delta(v) \geqslant \eta$.

4. Sketched proof of of Theorem 3. In order to prove Theorem 3 one must do the following. In the proof of Theorem 1 instead of the inequality (6) of Lemma 1 one should prove

$$\mu\left(I_\nu^{(n-h(n))}\bigcap A_n\right) \leqslant 4\delta(n)\mu\left(I_\nu^{(n-h(n))}\right),$$

where $I_\nu^{(n-h(n))}$ is from partition (7). Then under the condition

$$\mu\left(I_{\nu'}^{(\tau_0-h(\tau_0))}\cap A_{\tau_0}\right) \geqslant \eta\mu\left(I_\nu^{(\tau_0-h(\tau_0))}\right) > 0$$

one should prove instead of the inequality (11) of Lemma 2 the following inequality:

$$\mu\left(I_\nu^{(\tau_0)}\bigcap\left(\bigcap_{j\leqslant\tau} A_j^c\right)\right) \geqslant \left(1-\frac{4}{\eta}\sum_{v=\tau_1}^{\tau-1}\delta(v)\right)\times\mu\left(I_\nu^{(\tau_0)}\right).$$

It means that in each interval of the form $I_\nu^{(\tau_0)}$ there exist not less than

$$N = \frac{\mu\left(I_\nu^{(\tau_0)}\bigcap\left(\bigcap_{j\leqslant\tau} A_j^c\right)\right)}{\mu\left(I_{\nu'}^{(\tau)}\right)} \geqslant \eta 2^{l_\tau - l_{\tau_0}}$$

pairwise disjoint subintervals of the form $I_{\nu'}^{(\tau)}$. Then as in [8] one should take into account the convergence of (3) and apply the following well-known result:

Theorem (Eggleston [9]). *Let for every k we have a set $A_k = \bigsqcup_{i=1}^{R_k} I_k(i)$ where $I_k(i)$ are segments of real line of length $|I_k(i)| = \Delta_k$. Let each interval $I_k(i)$ has exactly $N_{k+1}>1$ pairwise disjoint subintervals $I_{k+1}(i')$ of length Δ_{k+1} from the set A_{k+1}. Let $R_{k+1}=R_k\cdot N_{k+1}$. Suppose $0<\nu_0\leqslant 1$and for every $0<\nu<\nu_0$ the series $\sum_{k=2}^\infty \frac{\Delta_{k-1}}{\Delta_k}(R_k(\Delta_k)^\nu)^{-1}$ converges. Then the set $A=\bigcap_{k=1}^\infty A_k$ has Hausdorff dimension* $\mathrm{HD}(A)\geqslant\nu_0$.

6. Examples. Note that the proof of Theorem 1 follows directly the arguments by Y.Peres and W. Schlag from [7]. The author in [10] (following Peres-Schlag's arguments) established for lacunary sequence $\{t_n\}$ under condition

$$\frac{t_{j+1}}{t_j} \geqslant 1 + \frac{1}{M}, \ \ \forall j \in \mathbb{N}.$$

the existence of a real number α such that

$$||\alpha t_j|| \geqslant \frac{1}{2^{11} M \log M}, \ \ \forall j \in \mathbb{N}.$$

. We consider some examples with sublacunary sequences below.

A. Sublacunary sequences. Let $\{t_n\}_{n=1}^\infty$ satisfy the condition

$$\frac{t_{n+1}}{t_n} \geqslant 1 + \frac{\gamma}{n^\beta}, \ \ 0 \leqslant \beta < 1, \ \ \gamma > 0. \tag{13}$$

We take $\eta < 1$ close to 1 and

$$h(n) = \lfloor c_1 n^\beta \log(n+c_2) \rfloor, \quad \delta(n) = \frac{(1-\beta)(1-\eta)\eta}{2^5 c_1 (n+c_2)^\beta \log(n+c_2)}, \tag{14}$$

Here large positive constants c_1, c_2 (depending on β and η) should be defined in the following way. In our situation under condition $n > h(n)$ for $\gamma_1 < \gamma_1$ one has

$$\frac{t_n}{t_{n-h(n)}} \geqslant \prod_{j=n-h(n)}^{n-1} \left(1 + \frac{\gamma}{j^\beta}\right) \geqslant \exp\left(\sum_{j=n-h(n)}^{n-1} \log\left(1+\frac{\gamma}{j^\beta}\right)\right) \geqslant \exp\left(\omega \frac{h(n)}{n^\beta}\right) \geqslant (n+c_2)^{\omega c_1}$$

with $\omega = \omega(\beta, \gamma_1)$. Let $c_1 = c_1(\beta, \eta)$ be a large positive constant such that for all real $y \geqslant 2$ we have

$$y^{\omega c_1} \geqslant \frac{2^5 c_1 y^\beta \log y}{(1-\beta)(1-\eta)\eta}.$$

Then

$$\frac{t_n}{t_{n-h(n)}} \geqslant (n+c_2)^{\omega c_1} \geqslant \frac{2^5 c_1 (n+c_2)^\beta \log(n+c_2)}{(1-\beta)(1-\eta)\eta} = \frac{1}{\delta(n)} \geqslant \frac{1}{\delta(n-h(n))}$$

and the condition (i′) of Theorem 2 is satisfied.

So we have c_1 fixed and then we define c_2. Let $c_2 = c_2(\beta)$ be a large positive constant such that

$$\max_{n \in \mathbb{N}} \frac{4c_1 \log(n+c_2)}{(n+c_2)^{1-\beta}} \leqslant 1, \tag{15}$$

$$h\left(\frac{1}{2^5\delta(0)}\right) = \left\lfloor c_1 \left(\frac{c_1 c_2^\beta \log c_2}{(1-\beta)(1-\eta)\eta}\right)^\beta \log\left(\frac{2^2 c_1 c_2^\beta \log c_2}{1-\beta} + c_2\right)\right\rfloor \leqslant \frac{1}{2^5\delta(0)} = \frac{c_1 c_2^\beta \log c_2}{(1-\beta)(1-\eta)\eta}. \tag{16}$$

$$\min_{y \geqslant 1} \left((1-\beta)\log(y+c_2) - \frac{y}{y+c_2}\right) > 0 \tag{17}$$

Then from (15) it follows that $\frac{h(n)}{n+c_2} \leqslant \frac{1}{2}$ and for $n > h(n)$ we have

$$\sum_{v=n-h(n)+1}^{n-1} \delta(v) \leqslant \frac{(1-\beta)(1-\eta)\eta}{2^5 c_1 \log(n-h(n)+c_2)} \sum_{v=n-h(n)+1}^{n-1} \frac{1}{v^\beta} \leqslant (1-\eta)\eta \times \frac{n^{1-\beta} - (n-h(n))^{1-\beta}}{2^4 c_1 \log(n-h(n)+c_2)} \leqslant$$

$$\leqslant \frac{(1-\eta)\eta h(n)}{2^4 c_1 n^\beta \log(n-h(n)+c_2)} \leqslant \frac{(1-\eta)\eta \log(n+c_2)}{2^3 \log(n-h(n)+c_2)} =$$

$$= \frac{(1-\eta)\eta}{2^3} \times \frac{\log(n+c_2)}{\log(n+c_2) + \log(1-\frac{h(n)}{n+c_2})} \leqslant \frac{(1-\eta)\eta}{2^3} \times \frac{\log(n+c_2)}{\log(n+c_2) - \log 2} \leqslant \frac{(1-\eta)\eta}{4}.$$

So the condition (ii′) of Theorem 2 is satisfied.

Moreover for the value $n_0 = n_0(\beta, c_1, c_2) = \max\{n \in \mathbb{N} : \; n \leqslant h(n)\}$ from (16) it follows that $n_0 \leqslant \frac{1}{2^5\delta(0)}$ and the condition (iii′) of Theorem 2 is satisfied also.

Also we must note that if $y \geqslant 1$ and $y > h(y) \geqslant c_1 y^\beta \log(y+c_2)$ then the function $y - c_1 y^\beta \log(y+c_2)$ is increasing as from (17) it follows that

$$(y - c_1 y^\beta \log(y+c_2))' = 1 - \beta c_1 y^{\beta-1} \log(y+c_2) - \frac{c_1 y^\beta}{y+c_2} = c_1 y^{\beta-1}\left((1-\beta)\log(y+c_2) - \frac{y}{y+c_2}\right) > 0.$$

Now we have checked all the conditions of Theorem 2. It follows that the set

$$\mathcal{B} = \{ \ \alpha \in [0,1] : \ \exists \varkappa > 0 \ \ \forall n \in \mathbb{N} \ \ ||t_n \alpha|| > \frac{\varkappa}{n^\beta \log(n+1)} \ \}$$

is nonempty (obviously, uncountable).

Note that the set $\{n \in \mathbb{N} : \ n \leqslant h(n)\}$ is finite. Hence we can construct a sequence of naturals $\{n_k\}$ satisfying (2).

If it happens that in addition to (13) we have

$$\frac{t_{n+1}}{t_n} \leqslant 1 + \frac{\gamma_2}{n^\beta} \tag{18}$$

with some $\gamma_2 > \gamma$ then for the sequence $\{n_k\}$ we get $t_{n_k} \leqslant t_{n_{k-1}} n^{\gamma_3}$ and $k \leqslant \gamma_4 n_k^{1-\beta}$ with positive $\gamma_{3,4}$. Now

$$\frac{1}{\eta^k} \cdot \frac{t^\nu_{n_k}}{t_{n_{k-1}}} \ll \frac{1}{e^{\gamma_5 n_k^{1-\beta}}} \cdot \frac{1}{\eta^k} \ll \frac{1}{(e^{\gamma_5}\eta^{\gamma_5})^{n^{1-\beta}}}$$

(here all constants γ_j do not depend on η) and for η close to 1 the series (3) converges. From Theorem 3 it follows that the set $\mathcal{B}$ has Hausdorff dimension equal to 1. We should note that it is possible to choose function $h(n)$ (actually in the same manner as it was done in [8]) to satisfy the conditions of Theorem 3 without additional assumption (18) on the rate of growth of the sequence t_n.

We should note that it would be interesting to investigate *winning* properties of the considered sets (for the definition of winning sets see [11],[12], for some partial results see [13]).

B. Subexponentional sequences. Let $\{t_n\}_{n=1}^\infty$ satisfy the condition

$$\gamma_1 \exp(n^\beta) \leqslant t_n \leqslant \gamma_2 \exp(n^\beta), \ \ 0 < \beta < 1, \ \ \gamma_{1,2} > 0. \tag{19}$$

Then by the same reasons (as in example **A**) we have that the Hausdorff dimension of the set

$$\{ \ \alpha \in [0,1] : \ \exists \varkappa > 0 \ \forall n \in \mathbb{N} \ ||t_n \alpha|| > \frac{\varkappa}{n^{1-\beta} \log(n+1)} \ \}$$

is equal to 1.

C. Fürstenberg's sequence. Consider the set of naturals of the form $2^n 3^m$ and let the sequence

$$s_1{=}1, s_2{=}2, s_3{=}3, s_4{=}4, s_5{=}6, s_6 = 8, \ldots$$

performs this set as an increasing sequence. Fürstenberg [14] (see also [15]) proved that for any irrational α the set of fractional parts $\{2^n 3^m \alpha\}$ is dense in $[0,1]$. Hence

$$\liminf_{n \to \infty} ||s_n \alpha|| {=} 0.$$

We should note that we no nothing about the rate of convergence to zero here. Obviously for $\alpha = 1/5$ one has

$$||s_n/5|| \geqslant 1/5.$$

But $1/5$ is a rational number.

The sequence $\{s_n\}$ satisfy (19) with $\beta = 1/2$. So from example **B** it follows that Hausdorff dimension of the set

$$\{ \ \alpha \in [0,1] : \ \exists \varkappa > 0 \ \forall n \in \mathbb{N} \ ||s_n \alpha|| > \frac{\varkappa}{\sqrt{n} \log(n+1)} \ \}$$

is equal to 1.

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author: Nikolay Moshchevitin

e-mail: moshchevitin@mech.math.msu.su, moshchevitin@rambler.ru